Zariski Topology

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Suppose R is a ring(not necessarily unital nor abelian) and denote all (proper) prime ideals of R by $\mathrm{Prim}(R)$. For each subset S of R, define

$$\vee(S) := \{P \in \mathrm{Prim}(R) | S \subset P\}.$$

Theorem 1. *$\{\mathrm{Prim}(R) \setminus \vee(S) | S \subset R\}$ forms a topology of R.*

Proof. In fact, (a)$\vee\{0\} = \mathrm{Prim}(R)$, $\vee(R) = \emptyset$.

(b)$\cap_\lambda \vee (S_\lambda) = \vee(\cup_\lambda S_\lambda), S_\lambda \subset R$. This can be seen straightforwardly by definition without using any property of an ideal.

(c)$\vee(S_1) \cup \vee(S_2) = \vee(S_1 S_2), S_1, S_2 \subset R$.

If $P \in \vee(S_1) \cup \vee(S_2)$, then $S_1 \subset P$ or $S_2 \subset P$, thus $S_1 S_2 \subset P$ since P is an ideal, and hence $\vee(S_1) \cup \vee(S_2) \subset \vee(S_1 S_2)$.

Suppose $P \in \vee(S_1 S_2)$,i.e., $S_1 S_2 \subset P$. If $S_1 \not\subset P$ and $S_2 \not\subset P$, then there exists some $s_1 \in S_1 \setminus P$ and $s_2 \in S_2 \setminus P$, thus $s_1 s_2 \notin P$ since the complement of a prime ideal is multiplication closed, a contradiction. Hence $\vee(S_1 S_2) \subset \vee(S_1) \cup \vee(S_2)$. □

We call this topology *Zariski topology* of $\mathrm{Prim}(R)$, and call $\mathrm{Prim}(R)$ with Zariski topology the *spectrum* of R.

We can verify the following properties directly:

$$\cap \vee S \supset S, S \subset R;$$
$$\vee \cap \mathcal{S} \supset \mathcal{S}, \mathcal{S} \subset \mathrm{Prim}(R);$$
$$\vee S_1 \supset \vee S_2, S_1 \subset S_2;$$
$$\cap \mathcal{S}_1 \supset \cap \mathcal{S}_2, \mathcal{S}_1 \subset \mathcal{S}_2.$$

(For the empty set $\emptyset$ of $\mathrm{Prim}(R)$,define $\cap\emptyset = R$.)

Property 2. *For each subset $\mathcal{S}$ of $\mathrm{Prim}(A)$, $\vee \cap \mathcal{S}$ is the closure of $\mathcal{S}$.*

Proof. Suppose $\mathcal{S} \subset \vee S$ for some $S \subset R$, then $\cap \mathcal{S} \supset \cap \vee S$, and thus $\vee \cap \mathcal{S} \subset \vee \cap \vee S \overset{\text{see here}}{=\!=} \vee S$. Therefore, $\vee \cap \mathcal{S}$ is the smallest closed set in $\mathrm{Prim}(R)$ that contains $\mathcal{S}$. □

For each semiprime ideal P of R,

$$\cap \vee P = P.$$

Recall that an ideal P of a ring R is called *semiprime* if P is the intersection of some prime ideals of R.

If R is a unital abelian ring, I is an ideal of R, then

$$\cap \vee I = \sqrt{I},$$

where $\sqrt{R} = \{r \in R | r^n \in I \text{ for some positive interger } n\}$.