Compactness of integral operators

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Let X and Y be locally compact Hausdorff spaces. For every function f on X and g on Y, we can define a function $f \otimes g$ on $X \times Y$ by

$$f \otimes g(x, y) = f(x)g(y).$$

If $f \in C_c(X)$ and $g \in C_c(Y)$, then $f \otimes g$ is a continous function on $X \times Y$ with a support contained in $\operatorname{supp} f \times \operatorname{supp} g$. Take distinct points (x, y) and (x', y') in $X \times Y$. We may assume that $x \neq x'$, so there is a $f \in C_c(X)$ such that $f(x) = 1$ and $f(x') = 0$. Take another function $g \in C_c(Y)$ such that $g(y) = g(y') = 1$, then $f \otimes g$ is a function in $C_c(X \times Y)$ such that $f \otimes g(x, y) = 1$ and $f \otimes g(x' \times y') = 0$, which follows that the subset

$$\mathcal{P} := \{f \otimes g | f \in C_c(X), g \in C_c(Y)\}$$

separates points of $X \times Y$ and vanishes nowhere. By locally compact version of Stone–Weierstrass theorem, we have that $\mathcal{P}$ is a dense subset of $C_c(X \times Y)$ in the topology of uniform convergence.

Suppose $h \in C_c(X \times Y)$, U and V be precompact open sets in X and Y that contain the projections S_X and S_Y of the support S of h onto X and Y respectively. Given any $\epsilon > 0$, the above argument shows that there are finite pairs of $f_i \in C_c(X)$ and $g_i \in C_c(Y)$ such that $\|h - \sum_i f_i \otimes g_i\|_\infty < \epsilon$. Moreover, we can require that f_i and g_i are supported in U and V respectively, since we can replace f_i and g_i by $f_i\phi$ and $g_i\psi$ where ϕ and ψ are functions in $C_c(X, [0, 1])$ and $C_c(Y, [0, 1])$ that are equal to 1 on S_X and S_Y and vanish outside U and V respectively. Note that $\|h - \sum_i (f_i\phi) \otimes (g_i\psi)\|_\infty = \|\phi \otimes \psi(h - \sum_i f_i \otimes g_i)\|_\infty \le \|h - \sum_i f_i \otimes g_i\|_\infty$.

Suppose μ is a Radon measure on $X \times Y$ and $h_0 \in L^p(X \times Y, \mu)$ $(1 \le p < \infty)$. For every $\eta > 0$, as $C_c(X \times Y)$ is dense in $L^p(X \times Y, \mu)$, there is a $h \in C_c(X \times Y)$ such that $\|h - h_0\|_p < \eta$. Assume f_i and g_i are functions like above, then

$$\begin{aligned}
\|h_0 - \sum_i f_i \otimes g_i\|_p &\le \|h_0 - h\|_p + \|h - \sum_i f_i \otimes g_i\|_p \\
&\le \eta + \|h - \sum_i f_i \otimes g_i\|_\infty \mu(U \times V)^{1/p} \\
&\le \eta + \epsilon\mu(U \times V)^{1/p}.
\end{aligned}$$

Take $\epsilon = \eta/\mu(U \times V)^{1/p}$, then we have that $\|h_0 - \sum_i f_i \otimes g_i\|_p < 2\eta$. This shows that $\mathcal{P}$ is dense in $L^p(X \times Y, \mu)$ $(1 \leq p < \infty)$.

Theorem. *Let X and Y be locally compact Hausdorff spaces, μ and ν be Radon measures on X and Y, respectively. If k is a measurable function on $X \times Y$ such that $\sup_{x \in X}\{\int_Y |k(x,y)|^q d\nu(y)\} < \infty$ $(1 < q < \infty)$, then $(Kf)(x) = \int_Y k(x,y)f(y)d\mu(y)$ defines a compact operator K from $L^p(Y,\nu)$ to $L^p(X,\mu)$ $(1/p + 1/q = 1)$.*

Proof. When $k = f \otimes g$ where $f \in C_c(X)$ and $g \in C_c(Y)$, then $\int_Y f(x)g(y)h(y)\mathrm{d}\nu(y) = f(x)\int_Y g(y)h(y)\mathrm{d}\nu(y)$. It shows that K is a rank-one operator. When k is a linear combination of functions $f \otimes g$, then K is a finite-rank operator. When k is given by the above theorem, we can approximate k by functions in the linear span of $\{f \otimes g | f \in C_c(X), g \in C_c(Y)\}$ in the topology of $L^q(X \times Y, \mu \times \nu)$. The condition $\sup_{x \in X}\{\int_Y |k(x,y)|^q d\nu(y)\} < \infty$ implies that the integral operator induced by k is bounded can be approximated by that induced by functions in the linear span of $\{f \otimes g | f \in C_c(X), g \in C_c(Y)\}$ and hence is compact. □

Detail Checking

1. A condition for $C_c(X)$ being dense in $L^p(X,\mu)$: μ is a Radon measure. (see Proposition 7.9, p217, Folland's Real Analysis)

2. todo... Check conditions for product measure.

Appendix

1. **Stone–Weierstrass theorem (locally compact spaces).**

> Suppose$X$ is a locally compact Hausdorff space and A is a subalgebra of $C_0(X,\mathbb{R})$. Then A is dense in $C_0(X,\mathbb{R})$ (given the topology of uniform convergence) if and only if it separates points and vanishes nowhere (i.e., for all $x \in X$ there is a $f \in A$ such that $f(x) \neq 0$).

2. There is a similar result in Exercise 6, p177, Conway's functional analysis.