Approximation by simple functions on a product σ -algebra

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Suppose (X, Ω_X) and (Y, Ω_Y) are measurable spaces. Denote by $\Omega_X \otimes \Omega_Y$ the σ-algebra on $X \times Y$ generated by sets of the form $A \times B$ where $A \in \Omega_X$ and $B \in \Omega_Y$. Let μ_X and μ_Y be measures on (X, Ω_X) and (Y, Ω_Y), respectively. It is easy to varify that

$$\|f\|_{p,q} := \left(\int_X \left(\int_Y |f(x,y)|^q \mathrm{d}\mu_Y(y)\right)^{p/q} \mathrm{d}\mu_X(x)\right)^{1/p}$$

defines a seminorm on a measurable functions with $\|f\|_{p,q} < \infty$ on $\Omega_X \times \Omega_Y$ for $p, q \in (1, \infty)$.

Lemma 1. *Let (X, Ω_X, μ_X) and (Y, Ω_Y, μ_Y) be finite measurable spaces. Denote by $\mathcal{F}$ the linear span of characteristic functions of $A \times B$ where $A \in \Omega_X$ and $B \in \Omega_Y$. For any characteristic function χ_C of a set $C \in \Omega_X \otimes \Omega_Y$, there is a sequence (s_n) in the set $\mathcal{F}$ that converges to χ_C in the seminorm $\|\cdot\|_{p,q}$ $(p, q \in (1, \infty))$.*

Proof. Let

$$\mathscr{A} := \{C \in \Omega_X \otimes \Omega_Y : \exists s_n \in \mathcal{F} \text{ s.t. } s_n \xrightarrow{\|\cdot\|_{p,q}} \chi_C\}.$$

One can check that it contains all sets of the form $A \times B$ where $A \in \Omega_X$ and $B \in \Omega_Y$. To prove the lemma, it suffices to show that $\mathscr{A}$ is a σ-algebra, i.e., $\mathscr{A}$ posses the following properties:

(a) $\mathscr{A}$ is closed under intersection.

Let $C, C' \in \mathscr{A}$ and $\epsilon > 0$. Take $s \in F$ such that $\|\chi_C - s\|_{p,q} < \epsilon$ and $s' \in F$ such that $\|\chi_{C'} - s'\|_{p,q} < \epsilon/(\|\chi_s\|_\infty + 1)$. Then

$$\begin{aligned}
&\|\chi_{C\cap C'} - ss'\|_{p,q} \\
&= \|(\chi_C - s)\chi_{C'} + s(\chi_{C'} - s')\|_{p,q} \\
&\leq \|\chi_C - s\|_{p,q}\|\chi_{C'}\|_\infty + \|s\|_\infty\|\chi_{C'} - s'\|_{p,q} \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Therefore, $C \cap C' \in \mathscr{A}$.

(b) $C, C' \in \mathscr{A}$ and $C \subset C'$ implies $C' \setminus C \in \mathscr{A}$.

This property follows from $\chi_{C' \setminus C} = \chi_{C'} - \chi_C$.

(c) If (C_n) is an increasing sequence in $\mathscr{A}$, then $C \coloneqq \bigcup_{n=1}^{\infty} C_n \in \mathscr{A}$.

Since χ_{C_n} converges to χ_C pointwise on $X \times Y$, $\int_Y |\chi_{C_n}(x,y) - \chi_C(x,y)|^q \mathrm{d}\mu_Y(y)$ decreases to zero for all $x \in X$. Therefore,

$$\begin{aligned}&\lim_n \int_X (\int_Y |\chi_{C_n} - \chi_C|^q(x,y)\mathrm{d}\mu_Y(y))^{p/q}\mathrm{d}\mu_X(x)\\ =&\int_X (\lim_n \int_Y |\chi_{C_n} - \chi_C|^q(x,y)\mathrm{d}\mu_Y(y))^{p/q}\mathrm{d}\mu_X(x)\\ =&0,\end{aligned}$$

that is, $\|\chi_{C_n} - \chi_C\|_{p,q} \to 0$ as $n \to \infty$. □

Theorem 2. *Let (X, Ω_X, μ_X) and (Y, Ω_Y, μ_Y) be σ-finite measure spaces. Denote by $\mathcal{F}$ the linear span of characteristic functions of $A \times B$ where $A \in \Omega_X$ and $B \in \Omega_Y$ satisfying $\mu_X(A)\mu_Y(B) < \infty$.*
(a) For any characteristic function χ_C of a set $C \in \Omega_X \otimes \Omega_Y$, there is a sequence (s_n) in the set $\mathcal{F}$ that converges to χ_C in the seminorm $\|\cdot\|_{p,q}$ $(p, q \in (1,\infty))$.
(b) For any measurable function f on $X \times Y$ with $\|f\|_{p,q} < \infty$, there is a sequence (s_n) in the set $\mathcal{F}$ that converges to f in the seminorm $\|\cdot\|_{p,q}$ $(p, q \in (1,\infty))$.

Proof. Let Z_n be an increasing sequence of sets in $\Omega_X \otimes \Omega_Y$ with $\mu_X \times \mu_Y(Z_n) < \infty$ and $X \times Y = \bigcup_{n=1}^{\infty} Z_n$. As $\chi_C - \chi_{C \cap Z_n}$ decreases to zero pointwise on $X \times Y$ as $n \to \infty$, there is some n such that $\|\chi_C - \chi_{C\cap Z_n}\|_{p,q} < \epsilon$. From the case of finite measure spaces, there is a $s_n \in \mathcal{F}$ vanishing outside Z_n such that $\|s_n - \chi_{C\cap Z_n}\|_{p,q} < \epsilon$. Therefore, $\|s_n - \chi_C\|_{p,q} < 2\epsilon$.

(b) Approximate f by simple functions and apply the result of (a). □

Remark 3. *A set $C \in \Omega_X \otimes \Omega_Y$ may not be approximated by rectangles in measure. For example, let $X = Y = \{0,1\}$ with the counting measure. Then the set $\{(0,0),(1,1)\}$ cannot be approximated by rectangles in measure.*