Notes on ZFC Axiomatic Language and Formal Proofs

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The Language of ZFC

Symbols

The basic symbols of the language are as follows:

- **Variables:** $x_1, x_2, \ldots$
- **Predicate symbols:**
  - Equality: $=$
  - Membership: $\in$
- **Logical connectives:**
  - Negation: $\neg$
  - Implication: $\to$
- **Quantifier:**
  - Universal quantifier: $\forall$

No function symbols or constant symbols are included in the basic language of ZFC.

Formulas

Formulas are defined inductively.

Atomic Formulas

The atomic formulas are:

- $x_1 = x_2$
- $x_1 \in x_2$

Compound Formulas

If φ and ψ are formulas, then the following are also formulas:

- $\neg\varphi$

- $\varphi \to \psi$
- $\forall x_n \varphi$

There are no other formulas beyond those generated by these rules.

Axioms of Predicate Logic

The deductive system for ZFC is based on a Hilbert-style axiom system for first-order predicate logic. The logical axioms include:

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
3. $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
4. $\forall x_i \varphi(x_i) \to \varphi(t)$, where the term t remains free after substitution for the free occurrences of x_i in φ. In particular, t must not become bound by a quantifier $\forall t$.
5. $\forall x_i(\varphi \to \psi) \to (\varphi \to \forall x_i \psi)$, provided that x_i does not occur free in φ.

These axioms govern the logical behavior of implication, negation, and universal quantification.

Formal Proofs

Definition of Proof

Let Γ be a set of formulas and φ a formula. We say that φ is **provable from** Γ, written

$$\Gamma \vdash \varphi,$$

if there exists a finite sequence of formulas

$$\varphi_1, \varphi_2, \ldots, \varphi_n$$

such that $\varphi_n = \varphi$, and for each index i, one of the following holds:

- φ_i is an axiom of predicate logic; or
- $\varphi_i \in \Gamma$; or
- there exist indices $j, k < i$ such that φ_k is of the form $\varphi_j \to \varphi_i$ (i.e., φ_i follows from φ_j and $\varphi_j \to \varphi_i$ by modus ponens); or
- there exists an index $j < i$ such that $\varphi_i = \forall x_n \varphi_j$ (universal generalization).