$L^\infty(X,\mu)$ may fail to be a von Neumann algebra

Chun Ding • 7 Nov 2023

After discussing here, we finally notice that $L^\infty(X,\mu)$ may fail to be a von Neumann algebra. Here is a counter example.

Let X be an uncountable set, μ be a counting measure, and Σ be the family of set A such that A or the complement of A is countable. Let S be a subset of X such that both itself and its complement are uncountable, then the bounded subset $\{\chi_A | A \text{ is a countable subset of } S\}$ in $L^\infty(X,\mu)$ has no least upper bound.